Strange Operator Equations. Use direct substitution and the given definitions to evaluate strange operator equations.

Ex: If a ∗ b = ab - 1 and if a ⊕ b = a + b - 1, what is the value of 4 ∗ [(6 ⊕ 8) ⊕ (3 ∗ 5)] ?

Soln: Evaluate by parts from the inside out. 6 ⊕ 8 = 6 + 8 - 1 = 13. 3 ∗ 5 = 3(5) - 1 = 14.
This results in 4 ∗ [13 ⊕ 14] = 4 ∗ [13 + 14 -1] = 4 ∗ 26 = 4(26) - 1 = 103 .

Set Theory: A **set** is a collection of things: $S = \{a, b, c, \ldots\}$. The things in the set are called the **elements** of the set. An **ordered set** is a collection of things in order: $O = \{a_1, a_2, \ldots, a_n\}$.

Ex: The set of single-digit, positive integers is $S = \{9,1,8,2,5,6,7,4,3\}$.

The ordered set of single-digit, positive integers is $O = \{1,2,3,4,5,6,7,8,9\}$.

The elements of *O* and the elements of *S* are the same: 1,2,3,4,5,6,7,8,9.

If you are given two sets then there are two common operations that can be performed on these sets:

Union: The **union** (join) of two sets *A* and *B*, written $A \bigcup B$, is a set containing every element of A **and** every element of *B*.

Ex: If $A = \{2,4,10\}$ then $S \bigcup A = \{9,1,8,10,2,5,6,7,4,3\}$ (*S* defined above).

Intersection: The **intersection** (both) of two sets *A* and *B*, written $A \bigcap B$, is a set containing all the elements which are simultaneously in **both** *A* and *B*.

Ex: $S \bigcap A = \{2,4\}$, where *A* and *S* are defined above.

Empty Set: The **empty** set contains **no** elements, denoted ϕ.

Ex: If $I = \{negative\ integers\}$ then $I \bigcap S = \phi$.

Ex: Notice that $X = \{0\}$ is not empty since *X* contains the element 0.

Complementary Set: The **complement** of a set Y is the set of all elements **not** in Y. This is often written $\overline{Y}$. The complement of a set *Y* in another set *Z* is often written $Z - Y$.

Ex: The complement of I in the set of all integers equals $\{0, \textit{positive integers}\}$.

Subset: A **subset** *B* of some given set *A* is some collection of elements all of which are elements of *A*. The **empty** set is always a subset of every set.

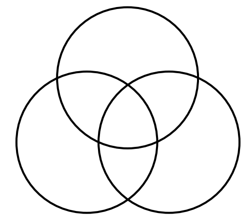
Ex: $\{2,4\}$ is a subset of *S* above.

Ex: $A \bigcap B$ is always a subset of both *A* and *B*.

Basic Rule: $\boxed{N(A) + N(B) = N(A \bigcup B) + N(A \bigcap B)}$ where **N(A)** denotes the number of elements of A.

Ex: The intersection of sets X and Y contains 5 elements. If X contains 9 elements, and Y contains 8 elements, how many elements are in the union of X and Y?

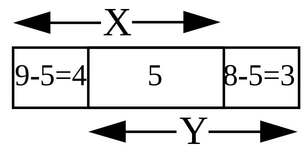
Soln: $N(X) + N(Y) = N(X \bigcup Y) + N(X \bigcap Y)$ so $9 + 8 = N(X \bigcup Y) + 5$ so
$N(X \bigcup Y) = 17 - 5 = 12$.



Venn Diagram: A pictorial way to solve logic problems.

Ex: Solve the above problem with a Venn diagram.

Soln: In the figure shown X is the long rectangle and Y is the other long rectangle. The total is $X \cup Y$ and the overlap is $X \cap Y$, so finding the numbers in each box, and then adding the three boxed numbers gives the answer as $4+5+3=12$. Other examples are in the solutions to the problems below.



Probability: Suppose that an event can happen in h ways and fail to happen in f ways, with all the $h+f$ ways equally likely.

Then the probability of the event **happening** is $\boxed{p = \frac{h}{h+f}}$,

and the probability of the event **not happening** is $\boxed{q = \frac{f}{h+f}}$.

Important: $\boxed{p+q=1}$, so $q = 1-p$ or $p = 1-q$.

Independent Events: Two or more events are said to be **independent** if the occurrence of any one of them does not affect the occurrence of any of the others.

Ex: A coin is tossed and comes up heads 4 times in a row. If the coin is honest then the probability of getting a head on the 5th toss is still $\frac{1}{2}$.

The probability that two or more **independent** events will happen is equal to the **product** of their separate probabilities.

Ex: The probability of getting tails on the 17th and 22nd tosses is $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$.

Dependent Events: Two or more events are said to be **dependent** if the occurrence of any one of the events affects the probability of occurrence of any of the others.

Ex: A box contains 3 white balls and 2 black balls. What is the probability of drawing 2 white balls in your first two draws with non-replacement?

$$p_1 = \frac{w}{b+w} = \frac{3}{2+3} = \frac{3}{5}. \quad p_2 = \frac{w_1}{b+w_1} = \frac{2}{2+2} = \frac{2}{4} = \frac{1}{2}. \quad p = p_1 \bullet p_2 = \left(\frac{3}{5}\right)\left(\frac{1}{2}\right) = \boxed{\frac{3}{10}}.$$

Mutually Exclusive Events: Two or more events are said to be **mutually exclusive** if the occurrence of any one of them prevents the occurrence of the others.

The probability of occurrence of several mutually exclusive events is the **sum** of the probabilities of the individual events.

Ex: What is the probability of getting heads at least twice in 5 honest coin tosses?

$$p(0\ heads) + p(1\ head) + p(\geq 2\ heads) = 1 = p_0 + p_1 + p_2 \text{ so } p_2 = 1 - p_0 - p_1.$$

$$p_0 = p(5\ tails) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}. \quad p_1 = 5 \bullet p(head) \bullet p(4\ tails) = 5 \bullet \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 = 5 \bullet \left(\frac{1}{2}\right)^5 = \frac{5}{32}.$$

(Any toss can be a head, but then all others must be tails, and the head can be any one of the 5 tosses.)

$$p_2 = 1 - \frac{1}{32} - \frac{5}{32} = 1 - \frac{6}{32} = 1 - \frac{3}{16} = \boxed{\frac{13}{16}}.$$

Probability from Areas: If an object must land somewhere in a given area, and has an equal chance of landing anywhere within the area, then the probability of landing within some sub-area of the given area is equal to the ratio of the sub-area to the total given area.

Ex: A dart has the same chance of landing anywhere within a rectangle. The probability of the dart landing in the upper third of the rectangle is $\frac{A/3}{A} = \frac{1}{3}$, where A is the area of the rectangle.

Probability 1 Homework

(NO CALCULATORS)

a) Let ∇ be defined as $\nabla(a,b) = \sqrt{a^2 + b^2}$ for all real numbers a and b. Find $\nabla(12,5)$.

b) If $A \Diamond B = \frac{2A-B}{2}$, what is the value of $(3 \Diamond 4) \Diamond 5$? Express your answer as a common fraction.

c) Set A has 7 members. Set B has 6 members. The union of set A and set B has 10 members. How many elements are there in the intersection of the two sets?

d) Of the 30 students in Ms. Smith's class, 14 are boys. There are 13 students who play a musical instrument and six of these are boys. How many girls in the class do not play a musical instrument?

e) Every member of a math club is taking algebra or geometry and 8 are taking both. If there are 17 taking algebra and 13 taking geometry, how many members are in the club?

f) Out of 22 students surveyed on ice cream flavors, 12 liked chocolate, 5 liked only strawberry, and 6 liked vanilla. If three liked chocolate and vanilla, how many students did not like any of these flavors?

g) Of 10 boxes, 5 contain pencils, 4 contain pens, and 2 contain both pens and pencils. How many boxes contain neither pens nor pencils?

h) In how many ways can 5 letters be mailed if there are 3 mailboxes available?

i) One ball is drawn at random from a box containing 3 red balls, 2 white balls, and 4 blue balls. Determine the probability that the one ball drawn is (1) red, (2) not red, (3) white, (4) red or blue.

j) One bag contains 4 white balls and 2 black balls; another bag contains 3 white balls and 5 black balls. If one ball is drawn from each bag, determine the probability that (1) both are white, (2) both are black, (3) 1 is white and 1 is black.

k) Determine the probability of throwing a total of 8 in a single throw with two dice, each of whose 6 faces are numbered from 1 to 6.

l) What is the probability of getting at least one 3 in two throws of a die?

m) The probability of Bob winning a game of chess against Mary is 1/3. What is the probability that Bob will win at least 1 game out of 3 games played?

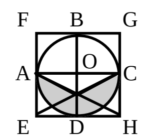
n) Three cards are drawn from a pack of 52, with replacement. What is the probability that (1) all are spades, (2) all are aces, (3) none are red cards? The 52 card pack contains 13 spades, 4 aces, 26 red cards.

o) The probability that Bob can solve a given problem is 4/5, that Mary can solve it is 2/3, and that Sam can solve it is 3/7. If all three try, what is the probability that the problem will be solved?

p) How many two-digit numbers can be formed with the digits 0, 3, 5, 7 if no repetition in any of the digits is allowed?

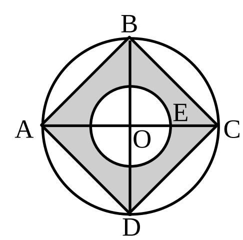
q) In how many ways can 3 girls and 3 boys be seated in a row if no two girls and no two boys are to occupy adjacent seats?

r) In the diagram, circle O is inscribed in the square *EFGH* with sides of length 12. The diameters $\overline{AC}$ and $\overline{BD}$ of circle O are perpendicular bisectors of $\overline{EF}$ and $\overline{FG}$ respectively. What is the area of the unshaded portion inside the outer square? Express your answer in terms of π.



s) An 8 inch by 10 inch rectangular picture is surrounded by a border that is 2 inches wide on all sides. Find the ratio of the area of the border to the area of the picture. Express your answer as a common fraction.

t) In the diagram both circles have center O, $\overline{AC}$ and $\overline{BD}$ are diameters of the larger circle, $\overline{AC} \perp \overline{BD}$, $AO = 2$, and $OE = 1$. Find the area of the shaded region. Express your answer as a common fraction in terms of π.



u) How many diagonals has an octagon? (A diagonal is any line **inside** the octagon between two vertices, not necessarily through the center.)

v) How many parallelograms are formed by a set of 4 parallel lines intersecting another set of 7 parallel lines?

w) How many ways can 12 books be divided between *A* and *B* so that one gets 9 and the other 3 books?

x) The probability that Charles will be alive 25 years from now is 3/7, and the probability that Alice will be alive 25 years from now is 4/5. Determine the probability that, 25 years from now, (1) both will be alive, (2) at least one will be alive, (3) only Charles will be alive.

y) One purse contains 5 dimes and 2 quarters, and a second purse contains 1 dime and 3 quarters. If a coin is taken from one of the two purses at random, what is the probability that it is a quarter?

z) What is the probability of landing a dart within the shaded square if the dart lands randomly anywhere within the large square? The length of the small square edge is 1/4 of the length of the large square edge.